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THE HUYGENS PROPERTY FOR THE HEAT EQUATION

BY

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Dedicated to R. P. Boas for his 65th birthday

ABSTRACT. This note summarizes several criteria which guarantee that a solution of the heat equation should also have the semigroup property described in equation (1.2) below. In particular, it corrects a mistake in an earlier proof of one of these.

1. **Introduction.** A function $u(x, t)$ belongs to class H in a strip $0 < t < c$ if it has continuous second order derivatives there and satisfies the heat equation

$$\partial^2 u/\partial x^2 = \partial u/\partial t. \tag{1.1}$$

It has the Huygens property or belongs to class H^0 there if, in addition, for every t and δ such that $0 < \delta < t + \delta < c$,

$$u(x, t + \delta) = k(x, t) * u(x, \delta) = \int_{-\infty}^{\infty} k(x - y, t)u(y, \delta)\, dy, \tag{1.2}$$

the integral converging absolutely. Here $k(x, t)$ is the source solution of equation (1.1),

$$k(x, t) = (4\pi t)^{-1/2}\exp\left[-x^2/(4t)\right], \qquad 0 < t.$$

It is a familiar fact that class H is larger than H^0 [1, p. 163]. Since relation (1.2) is so important in the applications of the heat equation it is evidently desirable to have convenient characterizations of class H^0.

In this note we give several criteria for admission to class H^0. In [1, p. 162] it was shown that any function defined by a convergent Poisson transform,

$$u(x, t) = k(x, t) * d\alpha(x) = \int_{-\infty}^{\infty} k(x - y, t)\, d\alpha(y), \tag{1.3}$$

belongs to H^0. A necessary and sufficient condition for the representation (1.3) with $\int_{-\infty}^{\infty} k(x, c)|d\alpha(x)| < \infty$ was stated in [2, p. 235]. However, in the proof there given there is a hiatus,[1] not easily filled. A primary purpose of the

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[1] It was shown that the function $w_h(x, t)$ belongs to H before applying Lemma 6.2 to it.

present note is to provide a valid (and simplified) proof of that theorem, using a different method.

2. The criteria. In this section we state four of the theorems to be proved.

THEOREM 2.1. *If* $u(x, t) \in H, 0 < t < c$, *and if*

$$|u(x,t)| \leqslant M \exp Ax^2, \qquad 0 < t < c \leqslant 1/(4A), \tag{2.1}$$

then

$$u(x,t) = k(x,t) * \varphi(x), \qquad 0 < t < c, \tag{2.2}$$

where

$$|\varphi(x)| \leqslant M \exp Ax^2, \qquad -\infty < x < \infty. \tag{2.3}$$

As noted in the introduction, it follows from (2.2) that $u(x, t) \in H^0$, $0 < t < c$.

THEOREM 2.2. *If* $u(x, t) \in H$, $0 < t < c$, *then a necessary and sufficient condition for the representation* (2.2), (2.3) *is that*

$$|u(x,t)| \leqslant Mk(x,t) * \exp Ax^2, \qquad 0 < t < c \leqslant 1/(4A). \tag{2.4}$$

The convolution (2.4) is equal to [1, p. 171]

$$k(x,t) * \exp Ax^2 = (1-4At)^{-1/2}\exp\left[Ax^2/(1-4At)\right], \qquad 0 < t < 1/(4A).$$

THEOREM 2.3. *If* $u(x, t) \in H, 0 < t < c$, *and if*

$$\int_{-\infty}^{\infty} |u(x,t)| \exp\left[-Ax^2\right] dx \leqslant M, \qquad 0 < t < c \leqslant 1/(4A), \tag{2.5}$$

then

$$u(x,t) = k(x,t) * d\alpha(x), \qquad 0 < t < c, \tag{2.6}$$

where

$$\int_{-\infty}^{\infty} k(x,c)|d\alpha(x)| < \infty. \tag{2.7}$$

THEOREM 2.4. *If* $u(x, t) \in H$, $0 < t < c$, *then a necessary and sufficient condition for the representation* (2.6), (2.7) *is that*

$$\int_{-\infty}^{\infty} |u(x,t)| k(x, c-t)\, dx \leqslant M, \qquad 0 < t < c. \tag{2.8}$$

We show at once that Theorem 2.1 is a corollary of Theorem 2.2. We have listed it first because it is usually the simplest condition to apply. We have only to note that

$$\exp Ax^2 < (1-4At)^{-1/2}\exp\left[Ax^2/(1-4At)\right], \qquad 0 < t < 1/(4A),$$

since $1 < 1/(1-4At)$ and $1 > (1-4At)^{1/2}$.

The example $u(x, t) = k(x, t) * \exp Ax^2$ shows that condition (2.1) is not necessary for the representation (2.2), (2.3). For, it has that representation; yet (2.1) is not satisfied for any constant M since

$$\lim_{|x|\to\infty} \exp\left[Ax^2/(1-4At) - Ax^2\right] = \infty, \qquad 0 < t < 1/(4A).$$

Theorem 2.2 is proved in [3, p. 206].

3. **The main result.** We proceed now to the proof of Theorem 2.4, which was Theorem 6.1 of [2]. The necessity of condition (2.8) is immediate. For, if

$$u(x, t) = \int_{-\infty}^{\infty} k(x-y, t)\, d\alpha(y),$$

this integral converges at $x = 0$, $t = c$ by (2.7) and, hence, defines a function of class H for $0 < t < c$ [1, p. 64]. Moreover,

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x, t)| k(x, c-t)\, dx &\leqslant \int_{-\infty}^{\infty} k(x, c-t)\, dx \int_{-\infty}^{\infty} k(x-y, t)|d\alpha(y)| \\ &= \int_{-\infty}^{\infty} |d\alpha(y)| \int_{-\infty}^{\infty} k(x, c-t) k(x-y, t)\, dx \\ &= \int_{-\infty}^{\infty} k(y, c)|d\alpha(y)| < \infty.\end{aligned}$$

We have used Fubini's theorem, applicable by virtue of (2.7). Thus (2.8) is satisfied.

For the converse we record a preliminary result.

LEMMA 3. *If* $a > 1$, $h > 0$, *then*

$$\exp\left[(|x|+h)^2\right] \leqslant \exp\left[ah^2/(a-1)\right]\exp ax^2, \qquad -\infty < x < \infty.$$

This is obvious if we note that the maximum value of the polynomial $(1-a)x^2 + 2xh + h^2$ for $0 \leqslant x \leqslant \infty$ is attained at $x = h/(a-1)$.

Now define

$$w_h(x, t) = \frac{1}{2h} \int_{x-h}^{x+h} u(y, t)\, dy, \qquad 0 < t < c.$$

Direct differentiation shows that $w_h(x, t) \in H$, $0 < t < c$. We show that $w_h(x, t) \in H^0$ also.

By (2.8) we have

$$|w_h(x,t)| \leqslant \frac{[\pi(c-t)]^{1/2}}{h}\int_{x-h}^{x+h}|u(y,t)|k(y,c-t)\exp\left[\frac{y^2}{4(c-t)}\right]dy$$

$$(3.1)\qquad \leqslant \frac{[\pi c]^{1/2}}{h}\exp\left[\frac{(|x|+h)^2}{4(c-t)}\right]\int_{-\infty}^{\infty}|u(y,t)|k(y,c-t)\,dy,$$

$$\frac{1}{2h}\int_{x-h}^{x+h}|u(y,t)|\,dy \leqslant \frac{M[\pi c]^{1/2}}{h}\exp\left[\frac{(|x|+h)^2}{4(c-t)}\right].$$

Applying Lemma 3 with some constant $a > 1$, we have

$$(3.2)\quad |w_h(x,t)| \leqslant \frac{M[\pi c]^{1/2}}{h}\exp\left[\frac{ah^2}{4(a-1)(c-t)}\right]\exp\left[\frac{ax^2}{4(c-t)}\right].$$

If we restrict t to the interval $0 < t < c/a$, then we may replace $c - t$ in the first exponent (3.2) by $\delta = c - (c/a)$ to obtain

$$(3.3)\quad |w_h(x,t)| \leqslant N\exp\left[\frac{ax^2}{4(c-t)}\right],\quad N = \frac{M[\pi c]^{1/2}}{h}\exp\left[\frac{ah^2}{4(a-1)\delta}\right].$$

The right-hand side of inequality (3.3) is less than

$$Nk(x,t) * \exp\left[\frac{ax^2}{(4c)}\right] = N[1-(at)/c]^{-1/2}\exp\left[\frac{ax^2}{4(c-at)}\right],$$

so that we may apply Theorem 2.2 with $A = a/(4c)$. We conclude that $w_h(x,t) \in H^0$ in $0 < t < c/a$ for every $a > 1$ and, hence, in $0 < t < c$.

From the definition of H^0 we have for $0 < \delta < t + \delta < c$,

$$(3.4)\quad \begin{aligned} w_h(x,t+\delta) &= \frac{1}{2h}\int_{-\infty}^{\infty}k(x-y,t)\,dy\int_{y-h}^{y+h}u(z,\delta)\,dz \\ &= \frac{1}{2h}\int_{-\infty}^{\infty}u(z,\delta)\,dz\int_{z-h}^{z+h}k(x-y,t)\,dy. \end{aligned}$$

To justify this change in the order of integration we have only to apply (3.1) and to note that

$$\int_{-\infty}^{\infty}k(x-y,t)\exp\left[\frac{(|y|+h)^2}{4(c-\delta)}\right]dy < \infty,$$

since $t < c - \delta$. Now allow h to approach zero in (3.4):

$$u(x,t+\delta) = \int_{-\infty}^{\infty}u(z,\delta)k(x-z,t)\,dz,$$

provided Lebesgue's theorem on dominant convergence is applicable. Set

$$
\begin{aligned}
R_h(x, z, t) &= \frac{1}{2h}\int_{z-h}^{z+h} k(x-y, t)\,dy \\
&= \frac{[\pi t]^{-1/2}}{4h}\int_{z-x-h}^{z-x+h} \exp\left[\frac{-y^2}{(4t)}\right] dy.
\end{aligned} \tag{3.5}
$$

We must show that

$$
\begin{aligned}
&|R_h(x,y,t)| \leqslant F(x,y,t), \qquad 0<h<1, \\
&\int_{-\infty}^{\infty} |u(z,\delta)| F(x,z,t)\,dz < \infty.
\end{aligned} \tag{3.6}
$$

For $|z-x| \leqslant 1$ it is sufficient to note that the integrand in (3.5) is bounded, so that we may take $F = [4\pi t]^{-1/2}$ there. For $|z-x|>1$ and $|y-z+x| \leqslant h<1$ we have $|y|>|z-x|-1$, so that we may take

$$
F(x,z,t) = [4\pi t]^{-1/2}\exp\left[\frac{-(|z-x|-1)^2}{(4t)}\right], \qquad |z-x|>1.
$$

Thus, $F(x,z,t)$ is independent of h, and (3.6) follows from

$$
\int_{|z-x|>1} |u(z,\delta)| k(z, c-\delta) \exp\left[\frac{z^2}{4(c-\delta)} - \frac{(|z-x|-1)^2}{4t}\right] dz < \infty. \tag{3.7}
$$

The exponential factor in (3.7) is a bounded function of z since $t < c-\delta$, so that hypothesis (2.8) establishes (3.7).

We have thus proved that $u(x,t) \in H^0$ in $0<t<c$. The final step, to establish the Poisson representation (2.6), (2.7), is accomplished by a standard application of the Helly selection theorem, as in [2, p. 238].

4. **Proof of Theorem 2.3.** This theorem seems not to be an immediate corollary of Theorem 2.4. However, slight modifications in the previous proof will give the desired result. Define $w_h(x,t)$ as before. Inequalities (3.1) and (3.3) become

$$
\begin{aligned}
|w_h(x,t)| &\leqslant \frac{1}{2h}\int_{x-h}^{x+h} |u(y,t)| \exp[-Ay^2] \exp Ay^2\,dy \leqslant \frac{M}{2h}\exp\left[A(|x|+h)^2\right] \\
&\leqslant (M/2h)\exp\left[Aah^2/(a-1)\right]\exp Aax^2, \qquad 0<t<c.
\end{aligned}
$$

We now apply Theorem 2.1 and conclude that $w_h(x,t) \in H^0$ on $0<t<c/a$ and, hence, on $0<t<c$. As before, equation (3.4) holds after noting that

$$
\int_{-\infty}^{\infty} k(x-y,t)\exp\left[A(|y|+h)^2\right] dy < \infty, \qquad A < 1/(4t).
$$

Inequality (3.7) becomes

$$\int_{|z-x|>1} |u(z, \delta)| \exp[-Az^2] \exp\left[Az^2 - \frac{(|z-x|-1)^2}{4t}\right] dz < \infty.$$

Allowing h to approach zero in (3.4) we again see that $u(x, t) \in H^0$ in $0 < t < c$, as desired. The proof is concluded as before.

We observe that condition (2.5) is not necessary for the representation (2.6), (2.7). For, if

$$d\alpha(x) = \exp[x^2/(4c)]/(1 + x^2)\, dx,$$

then

$$\int_{-\infty}^{\infty} k(x, c)\, d\alpha(x) = (4\pi c)^{-1/2} \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} < \infty.$$

But the integral (2.5), with $A = 1/(4c)$, becomes

$$\int_{-\infty}^{\infty} \exp \frac{[y^2/(4c)]}{1 + y^2}\, dy \int_{-\infty}^{\infty} k(x - y, t) \exp[-x^2/(4c)]\, dx$$

$$= [1 + (t/c)]^{-1/2} \int_{-\infty}^{\infty} \exp \frac{[ty^2/4c(t + c)]}{1 + y^2}\, dy = \infty, \qquad 0 < t.$$

This integral diverges for all $t > 0$, so that no constant M exists satisfying (2.5).

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